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Flow Behavior of Power Law Fluids in an Annulus

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Fredrickson and Bird (1958); Rotem (1962); McEachern (1966); Kozicki, Chou, and Tiu (1966); and Kozicki and Tiu (1971) have theoretically analyzed the fully developed laminar flow, and Tiu and Bhattacharya (1974) and Bhattacharya and Tiu (1974) have presented experimental results for developing and fully developed velocity and pressure profiles for inelastic power law fluids in an annulus. The pressure loss analysis of the entrance region flow and also the solution of energy equations require the knowledge of velocity profile in the developing boundary layer which is customarily chosen according to that in the fully developed region. Although El. Defrawi and Finlayson (1972) have pointed out the inappropriateness of the use of Newtonian velocity profile for non-Newtonian fluids in the boundary layer flow, Tiu and Bhattacharya (1973) had to opt for the second-order Newtonian profile for use in the momentum-energy integral technique, in view of the lengthy mathematics involved in the solution of the governing differential equations. Therefore, it seems desirable to obtain an approximate closed form of the velocity profile in fully developed flow satisfactorily agreeing with the exact analysis of Fredrickson and Bird (1958).

An annulus may be considered as the most general conduit of which a circular tube and parallel plate channel are two particular geometries. The local velocity profile for fluids, following a power law

$$\tau = K(-du/dr)^n \qquad (1)$$

flowing through these two extreme shaped conduits may be expressed as

$$u/u_m = \left[1 - Z^{\frac{n+1}{n}} \right] \qquad (2)$$

where Z is the dimensionless distance from zero shear position. On this basis the velocity profiles in the inner and outer flow regions of an annulus are proposed as

$$\frac{u_1}{u_m} = 1 - \left(\frac{r_m - r}{r_m - r_1} \right)^{\frac{n+1}{n}} ; \quad r_1 \leq r \leq r_m \qquad (3)$$

and

$$\frac{u_2}{u_m} = 1 - \left(\frac{r - r_m}{r_2 - r_m} \right)^{\frac{n+1}{n}} ; \quad r_m \leq r \leq r_2 \qquad (4)$$

From Equations (3) and (4), the average velocity in the annulus may be obtained as

$$\frac{<u>}{u_m} = \left[\frac{(n+1)}{(1+k)} \right] \left[\frac{(2n+1)(1+k) + 2n\lambda}{(3n+1)(2n+1)} \right] \qquad (5)$$

By substituting the values of τ_{w1} and τ_{w2} obtained from the velocity gradients at the inner and outer walls, respectively, in the overall momentum balance equation, the maximum velocity may be expressed as

$$u_m = \left(\frac{\Delta P\, r_2}{2KL} \right)^{1/n} \left[\frac{r_2{}^n (1-k^2)}{(1-\lambda)^{-n} + k(\lambda - k)^{-n}} \right]^{1/n} \left(\frac{n}{n+1} \right) \qquad (6)$$

From Equations (5) and (6), the volumetric flow rate Q may be expressed as

$$Q = \pi r_2{}^3 \left(\frac{\Delta P\, r_2}{2\,KL} \right)^{1/n} \Omega' \qquad (7)$$

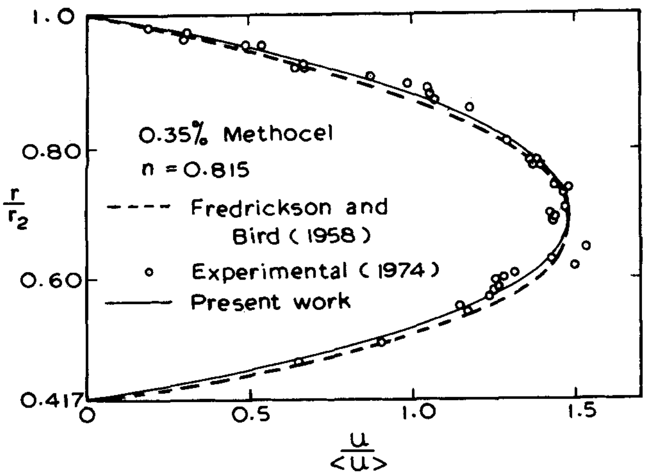

Fig. 1. Comparison of fully developed velocity profile for $k = 0.42$.

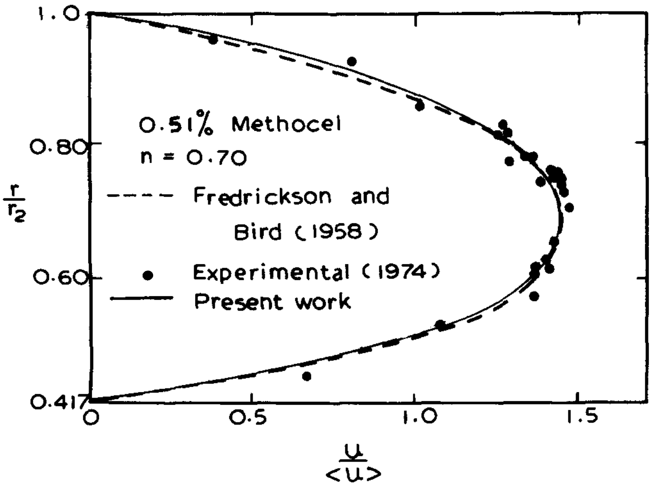

Fig. 2. Comparison of fully developed velocity profile for $k = 0.42$.

where

$$\Omega' = \left\{ \frac{n[(2n+1)(1+k) + 2n\lambda]}{(2n+1)(3n+1)(1+k)} \right\} \left[\frac{(1-k^2)^{n+1}}{(1-\lambda)^{-n} + k(\lambda - k)^{-n}} \right] \quad (8)$$

By defining the friction factor as

$$f = \frac{D_e \Delta P}{2L\rho \langle u \rangle^2} \quad (9)$$

where $D_e = 2r_2(1-k)$, its substitution in Equation (7) yields

$$f = 16/N'_{Re} \quad (10)$$

where

$$N'_{Re} = \frac{2^{3-n} D_e^{\,n} \langle u \rangle^{2-n} \rho}{K\left[\frac{1-k}{1-\lambda}\right]\left[\frac{(2n+1)(3n+1)(1-k^2)}{n(1-\lambda)[(2n+1)(1+k)+2n\lambda]}\right]^n} \quad (11)$$

The proposed approximate velocity profile and the experimental data point obtained by Tiu and Bhattacharya (1974) for 0.35% ($n = 0.815$) and 0.51% ($n = 0.7$) methocal solutions flowing through an annulus ($k = 0.417$) are compared with the theoretical profile from Fredrickson and Bird (1958) in Figures 1 and 2. The ratios u/u_m calculated from Equation (5) for various values of n and k are found to deviate by not more than 4% from the values obtained by Fredrickson and Bird. On the basis of the excellent agreement between the approximate and theoretical profiles, Equations (3) and (4) are recommended for the evaluation of velocity gradient and shear stress at the wall and also its application in the solution of the energy equation.

The viscosity of non-Newtonian fluids depends upon the shear stress, and therefore it can be defined either at the outer wall or at the inner wall of the annulus. Considering the outer flow region ($r_m \leq r \leq r_2$), the following expressions are obtained:

$$\frac{\langle u_2 \rangle}{u_m} = \left[\frac{(n+1)}{(2n+1)}\right]\left[\frac{(2n+1)+(4n+1)\lambda}{(3n+1)(1+\lambda)}\right] \quad (12)$$

and

$$\tau_{w2} = K\left\{ \frac{2(2n+1)(3n+1)(1+\lambda)^2}{n[(2n+1)+(4n+1)\lambda]} \frac{\langle u_2 \rangle}{De_2} \right\}^n \quad (13)$$

where $De_2 = 2r_2(1-\lambda^2)$. For Newtonian fluids, the above expression reduces to

$$\tau_{w2} = \mu\left[\frac{24\langle u_2 \rangle}{De_2}\frac{(1+\lambda)^2}{(3+5\lambda)}\right] \quad (14)$$

The rearrangement of Equation (13) results in

$$\tau_{w2} = K'\left[\frac{24\langle u_2 \rangle}{De_2}\frac{(1+\lambda)^2}{(3+5\lambda)}\right]^n \quad (15)$$

where

$$K' = K\left\{ \frac{(3+5\lambda)(2n+1)(3n+1)}{12n[(2n+1)+(4n+1)\lambda]} \right\}^n \quad (16)$$

Equation (15) reduces to conventional forms for tubes and parallel plates by setting $\lambda = 0$ and $\lambda = 1$, respectively.

Defining the pseudo shear or effective viscosity of the fluid at the outer wall of the annulus as

$$\mu_e = \frac{\tau_{w2}}{\left[\frac{24\langle u_2 \rangle (1+\lambda)^2}{De_2(3+5\lambda)}\right]} = K'\left[\frac{24\langle u_2 \rangle}{De_2}\frac{(1+\lambda)^2}{(3+5\lambda)}\right]^{n-1} \quad (17)$$

and the friction factor as

$$f_2 = \frac{2\tau_{w2}}{\rho\langle u_2 \rangle^2} \quad (18)$$

Equation (15) gives

$$f_2 = \frac{48}{N_{Re'2}}\frac{(1+\lambda)^2}{(3+5\lambda)} \quad (19)$$

where

$$N'_{Re2} = \frac{De_2^{\,n}\langle u_2 \rangle^{2-n}\rho}{K'\left[\frac{24(1+\lambda)^2}{(3+5\lambda)}\right]^{n-1}} = \frac{De_2\langle u_2 \rangle \rho}{\mu_e} \quad (20)$$

The Reynolds number N'_{Re2} uses the effective viscosity defined at the outer wall shear stress and reduces to conventional forms for power law fluids flowing through tubes and parallel plate channels.

By considering a numerical example of a power law fluid ($n = 0.3334$) flowing through an annulus ($k = 0.444$), as illustrated by Skelland (1967, p. 114), the

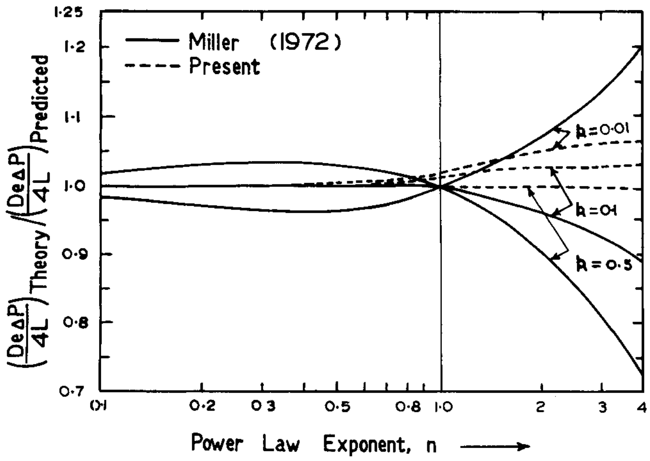

Fig. 3. Comparison of ($D_e\Delta P/4L$) from Fredrickson and Bird (1958) with those obtained from Miller (1972) and present methods.

following results for pressure drop are obtained using the available methods in the literature:

(a) Fredrickson and Bird (1958)	9 568	Pa/m
(b) Kozicki et al. (1971)	9 659	Pa/m
(c) Vaughn and Bergman (1966)	7 014	Pa/m
(d) Parallel plate approximation (1958)	9 636	Pa/m
(e) Miller (1972)	9 288	Pa/m
(f) Present method	9 754	Pa/m

As a result of the erroneous calculations, Skelland (1967, p. 115) reported a 2.17 fold overestimate of the pressure drop by parallel plate approximation compared to that obtained from the Fredrickson and Bird (1958) method, whereas the corrected result (a) shown as above leads to only a 1.0071 overestimate. The invalidity of power law model in an annular flow as suggested by Vaughn and Bergman (1966) does not agree with the experimental findings of Bhattacharya and Tiu (1974). The above example shows an excellent agreement between Fredrickson and Bird (1958), Kozicki et al. (1971), Miller (1972), and the present method. Vaughu and Bergman's (1966) procedure predicts a very low value of pressure drop.

Miller's (1972) approach, which is more general with respect to fluid model and duct geometry, is compared with the present approach for annular laminar flow of power law fluids in Figure 3, where the ratio of ($D_e\Delta P/4L$) deduced from the theoretical analysis of Fredrickson and Bird (1958) to that predicted from Miller (1972) and the present approach is plotted against the flow behavior index n for three annular aspect ratios, $k = 0.01$, 0.1, and 0.5. According to Miller's approach (1972), the ordinate of Figure 3 should depend weakly upon both aspect ratio k and flow behavior index n, but the marked effect of both variables k and n is obvious from Figure 3. The deviation is more pronounced for dilatent fluids ($n > 1$). The empirical closed form of the velocity profile presented in this paper does not exactly coincide with the Newtonian annular flow; however, deviation is not more than 2.2%. For both the pseudoplastic ($n < 1$) and dilatent ($n > 1$) fluids, taking the method of Fredrickson and Bird (1958) as the standard, the pressure drop predicted from Miller (1972) deviates more than that predicted from the present method.

NOTATION

De = hydraulic diameter, $2r_2(1 - k)$
De_2 = hydraulic diameter of the outer flow region, $2r_2(1 - \lambda^2)$
f = fanning friction factor
f_2 = friction factor defined by Equation (18)
k = aspect ratio of the annulus, r_1/r_2
K = consistency index for power law fluids
K' = fluid consistency index defined by Equation (16)
L = length of the annular test section
n, n' = flow behavior index
N'_{Re} = Reynolds number defined by Equation (11)
N'_{Re2} = Reynolds number defined by Equation (20)
ΔP = pressure loss
Q = volumetric rate
r = radial position in the annulus
r_1, r_2 = inner and outer radii of the annulus
r_m = radius of maximum velocity
u = local velocity
u_1, u_2 = velocity in the inner and outer flow zones of annulus
$\langle u \rangle$ = average velocity in the annulus
$\langle u_2 \rangle$ = average velocity in the outer flow zone of the annulus
u_m = maximum velocity
z = dimensionless distance

Greek Letters

λ = dimensionless radial position of maximum velocity, r_m/r_2
Ω' = function defined by Equation (8)
ρ = density
τ = shear stress
τ_{w1}, τ_{w2} = shear stress at the inner and outer walls of the annulus
μ = viscosity
μ_e = effective viscosity at the outer wall of the annulus as defined by Equation (17)

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